POISSON LIMITS FOR PAIRWISE AND AREA INTERACTION POINT PROCESSES

S. RAO JAMMALAMADAKA,* *University of California*

MATHEW D. PENROSE,** *University of Durham*

Abstract

Suppose n particles x_i in a region of the plane (possibly representing biological individuals such as trees or smaller organisms) have a joint density proportional to $\exp\{-\sum_{i<j} \phi(n(x_i - x_j))\}$, with ϕ a specified function of compact support. We obtain a Poisson process limit for the collection of rescaled interparticle distances as n becomes large. We give corresponding results for the case of several types of particles, representing different species, and also for the area-interaction (Widom–Rowlinson) point process of interpenetrating spheres.

Keywords: Area-interaction process; Gibbs distribution; limit laws; point process; spatial statistics; U-statistics

AMS 1991 Subject Classification: Primary 60G55
Secondary 60F05; 60D05

1. Introduction

Suppose the locations of n particles in d-dimensional space (such as biological individuals in a region of the plane or molecules in a region of space) are represented by points x_i in $\mathbb{R}^d$, with $d \geq 1$. A simple stochastic model for their positions has them independently distributed with common probability density function f (for example, f could be a measure of the richness of the soil at a given point on the ground). We call this the *null distribution.*

A more complex model is the *pairwise interaction* point process, also known as a form of *Gibbs distribution.* An energy function $\phi_n : \mathbb{R}^d \to \mathbb{R}$ is specified and the null density is weighted multiplicatively, by $\exp(-\phi_n(x_i - x_j))$ for each pair of distinct particles x_i, x_j, giving a joint density function of the form

$$f^{(n)}(x_1, \ldots, x_n) = z_n^{-1}\left(\exp\left\{-\sum_{i<j\leq n} \phi_n(x_i - x_j)\right\}\right)\prod_{i=1}^n f(x_i), \qquad (1.1)$$

with z_n a normalizing constant. For a discussion and bibliography on distributions of this form, see Diggle *et al.* [5], or Stoyan *et al.* [15, Section 5.5]. Here we consider only distributions which are conditioned to have n points.

We consider the case where $\phi_n(x)$ takes the form $\phi(n^2|x|^d)$, with ϕ a specified function of compact support. This means that as n becomes large the range of interaction becomes small in

such a way that the expected number of pairs of particles which interact, under the null distribution, tends to a finite limit. This corresponds to the 'sparseness' regime considered by Saunders *et al.* in [12, 13], who obtained limit distributions for the number of interacting pairs in the special case of a uniform null distribution and with certain specific step function forms for ϕ.

In this note we obtain limiting distributions for the set of small interparticle distances, for a large class of null distributions and functions ϕ (Theorems 3.1 and 3.2). We also consider the case where there are several types of particles (representing, for example, different biological species) with a different energy function for each pair of types of particles (see Theorem 4.1). Our proofs use Laplace functional methods and general results in the literature on U-statistics. Proposition 2.1 on generalized Gibbs distributions and Proposition 4.1 on multitype U-statistics are key steps in our proofs, and are also of some interest in their own right.

In Section 5 we apply the same methods to the *area-interaction point process*, conditioned to have n points. For our purposes, this is defined by a joint density of the form

$$f^{(n)}(x_1, \ldots, x_n) \propto (\exp\{-\gamma_n V_{r_n}(x_1, x_2, \ldots, x_n)\}) \prod_{i=1}^{n} f(x_i), \tag{1.2}$$

where $V_r(x_1, \ldots, x_n)$ denotes the volume of the union of balls of radius r centred at $x_1, \ldots, x_n$, and γ_n and r_n are parameters. This form of density (for $d = 2$, hence the nomenclature) was proposed by Baddeley and van Lieshout [2] in the context of spatial statistics, having previously appeared in the physics literature [17]. We investigate a limiting regime for the parameters (γ_n, r_n) which is analogous to that considered for pairwise interaction processes; there turns out to be an asymptotic equivalence with a pairwise interaction process with a specific non-step function form for ϕ.

Both for pairwise and area interactions, the 'sparse' limits considered are not particularly natural from the statistical physics perspective, but are quite reasonable in the representation of the locations, for example, of a rare plant or nesting sites of a rare bird.

2. A general result

We start by looking at *generalized Gibbs distributions*, by which we mean distributions for which some tractable null measure μ_n is weighted by a Radon–Nikodym derivative proportional to e^{-U}, where U is the value of an 'energy function' summed over the points of an induced point process. More formally, a generalized Gibbs distribution is a probability measure μ_n' defined by (2.1) below. We shall give a simple result (Proposition 2.1) about generalized Gibbs distributions which may by applicable in other settings than those considered here. See for example [16] for a variety of possible applications of models of this form.

Let S be a complete separable metric space, and let $\mathcal{B}$ denote the Borel σ-field on S. Let $\mathcal{M}$ be the space of locally finite counting measures on $(S, \mathcal{B})$. For $M \in \mathcal{M}$ and $g : S \to \mathbb{R}$ we use the inner product notation $\langle M, g \rangle$ for the integral $\int_S g \, dM$. Clearly M is identified with a set of points and $\langle M, g \rangle$ is the sum of the values of g at those points.

A *point process* on S is a random element ξ of $\mathcal{M}$. If λ is a locally finite measure on $(S, \mathcal{B})$, a *Poisson process* with mean measure λ is a point process for which the random variables $\langle \xi, \mathbf{1}_{A_i} \rangle$ are independent Poisson variables with respective means $\lambda(A_i)$ for any finite collection of disjoint sets $A_i \in \mathcal{B}$ with $\lambda(A_i) < \infty$. In terms of Laplace functionals, Poisson processes are characterized as follows (see [11, Proposition 3.6]):

Lemma 2.1. *Let ξ be a point process on S and let λ be a locally finite measure on $(S, \mathcal{B})$. Then ξ is a Poisson process with mean measure λ, if and only if for all measurable $g : S \to (-\infty, \infty]$ bounded below and of compact support,*

$$\mathrm{E}\exp(-\langle \xi, g\rangle) = \exp \int (\mathrm{e}^{-g(x)} - 1)\lambda(\mathrm{d}x).$$

Here and throughout this paper, we use the convention $\exp(-\infty) = 0$.

Our general result is in the following setting. For $n \in \mathbb{N}$, let $(\Omega_n, \mathcal{F}_n, \mu_n)$ be probability spaces and let $\xi_n : \Omega_n \to \mathcal{M}$ be measurable for $n = 1, 2, 3, \ldots$. Suppose $\phi : S \to (-\infty, \infty]$ is a measurable 'energy function' of compact support. The associated generalized Gibbs measures are probability measures μ_n' on $(\Omega_n, \mathcal{F}_n)$ defined by

$$\mu_n'(\mathrm{d}\omega) = z_n^{-1} \exp(-\langle \xi_n(\omega), \phi\rangle)\mu_n(\mathrm{d}\omega), \tag{2.1}$$

with z_n a normalizing constant. We write Δ_ϕ for the set of discontinuity points of ϕ.

We are interested in the asymptotic distributions of the ξ_n under the measures μ_n and μ_n'. For $A \subset S$ write ∂A for the boundary of A. If ξ and ξ_n $(n \in \mathbb{N})$ are point processes on S, we say the sequence ξ_n converges weakly to ξ, and write $\xi_n \Rightarrow \xi$, if the finite-dimensional distributions converge, i.e. if for any finite collection of bounded Borel sets A_i satisfying $\xi(\partial A_i) = 0$ almost surely, the joint distributions of $\xi_n(A_i)$ converge weakly to those of $\xi(A_i)$. This is equivalent to other definitions of weak convergence; see e.g. [4, Section 9.1]. Weak convergence of point processes is characterized in terms of Laplace functionals as follows [11, Proposition 3.19]:

Lemma 2.2. *Let ξ and ξ_n $(n = 1, 2, 3, \ldots)$ be point processes on S. Then $\xi_n \Rightarrow \xi$ if and only if $\mathrm{E}\exp(-\langle \xi_n, g\rangle) \to \mathrm{E}\exp(-\langle \xi, g\rangle)$ for all measurable $g : S \to [0, \infty]$ of compact support with $\langle \xi, \mathbf{1}_{\Delta_g}\rangle = 0$ almost surely.*

The main result of this section says that if under μ_n the point processes ξ_n are asymptotically Poisson, then under the generalized Gibbs distribution μ_n' given by (2.1), they are again asymptotically Poisson, with mean measure inflated locally by $\exp(-\phi(x))$:

Proposition 2.1. *Suppose ϕ is non-negative. Suppose that under μ_n the sequence of point processes ξ_n converges weakly to a Poisson process with locally finite mean measure λ satisfying $\lambda(\Delta_\phi) = 0$. Then under μ_n' the point processes ξ_n converge weakly to a Poisson process with mean measure λ' given by $\lambda'(\mathrm{d}x) = \mathrm{e}^{-\phi(x)}\lambda(\mathrm{d}x)$.*

Proof. Let E, E' denote expectation under μ_n, μ_n' respectively. By definition

$$z_n = \mathrm{E}\exp(-\langle \xi_n, \phi\rangle).$$

By Lemmas 2.1 and 2.2,

$$z_n \to \exp \int_S (\mathrm{e}^{-\phi(x)} - 1)\lambda(\mathrm{d}x) =: z.$$

Let $g : S \to [0, \infty]$ be measurable with $\lambda(\Delta_g) = 0$. By Lemma 2,

$$\mathrm{E}'\exp(-\langle \xi_n, g\rangle) = z_n^{-1} \int_S \exp(-\langle \xi_n(\omega), g + \phi\rangle)\mu_n(\mathrm{d}\omega)$$

$$\to z^{-1} \exp \int_S (\mathrm{e}^{-\phi(x)-g(x)} - 1)\lambda(\mathrm{d}x)$$

$$= \exp \int_S (\mathrm{e}^{-\phi(x)-g(x)} - \mathrm{e}^{-\phi(x)})\lambda(\mathrm{d}x) = \exp \int_S (\mathrm{e}^{-g(x)} - 1)\lambda'(\mathrm{d}x),$$

and the result follows by Lemmas 2.1 and 2.2.

3. Interpoint distances

Returning to the particle picture of Section 1, we extend Theorem 1.1 of [13] to general null densities f and non-negative energy functions ϕ. In what follows, $\delta(x)$ denotes the unit point measure at x and $c_d = \pi^{d/2}/\Gamma((d/2)+1)$, the volume of the unit ball in $\mathbb{R}^d$.

Theorem 3.1. *First let f be a bounded probability density function on $\mathbb{R}^d$, and let $\phi : \mathbb{R}_+ \to [0, \infty]$ be almost everywhere continuous with compact support. Suppose that for each n, $X_1^n, \ldots, X_n^n$ are d-dimensional random variables with joint probability density function*

$$f^{(n)}(x_1, \ldots, x_n) = z_n^{-1}\left(\exp\left\{-\sum_{i<j\leq n}\phi(n^2|x_i - x_j|^d)\right\}\right)\left(\prod_{i=1}^n f(x_i)\right) \qquad (3.1)$$

with z_n a normalizing constant. Then the sequence of 1-dimensional point processes

$$\xi_n := \sum_{i<j\leq n}\delta(n^2|X_i^n - X_j^n|^d) \qquad (3.2)$$

converges weakly to a Poisson process on $\mathbb{R}_+$ with mean measure λ_0 given by

$$\lambda_0(\mathrm{d}u) = \tfrac{1}{2}c_d\left(\int f^2\right)\mathrm{e}^{-\phi(u)}\,\mathrm{d}u. \qquad (3.3)$$

Proof. Let $\Omega_n = (\mathbb{R}^d)^n$ and let μ_n be the probability measure on Ω_n corresponding to independent variables with common density f, that is for $x = (x_1, \ldots, x_n) \in \Omega$ take $\mu_n(\mathrm{d}x) = \prod_{i=1}^n (f(x_i)\,\mathrm{d}x_i)$. If X_n is a random element of Ω_n with distribution μ_n, then $\xi_n(X_n)$ converges in distribution to a Poisson process on $\mathbb{R}_+$ of rate $c_d(\int f^2)/2$. See Silverman and Brown [14, Theorem C] or Jammalamadaka and Janson [7]. Applying Proposition 2.1 we get the result.

When ϕ takes negative values, this proof breaks down (see the remark following Theorem 1.1 of [13]), but can be retrieved when ϕ has a hard-core component, as we now show, extending Theorem 2.3 of [13] to general null densities and energy functions.

Theorem 3.2. *Suppose that $H > 0$ and $\phi : \mathbb{R}_+ \to [-H, +\infty]$ has a hard-core component, i.e. there exists $r_0 > 0$ such that $\phi(u) = +\infty$ for $0 \leq u \leq r_0^d$. Additionally assume there exists $r_1 \geq r_0$ such that $\phi(u) = 0$ for $u > r_1^d$. Suppose that for each n, $X_1^n, \ldots, X_n^n$ are d-dimensional random variables with joint probability density function given by (3.1). Then the sequence of point processes ξ_n defined by (3.2) converges weakly to a Poisson process with mean measure λ_0 given by (3.3).*

Proof. Let ξ be an inhomogeneous Poisson process on $\mathbb{R}_+$ with mean measure λ_0 given by (3.3). By Theorem 3.1, under μ_n we have the weak convergence of random variables

$$\exp(-\langle\xi_n, \phi\rangle) \Rightarrow \exp(-\langle\xi, \phi\rangle), \qquad (3.4)$$

but since ϕ is no longer assumed to be non-negative the corresponding convergence of expectations is not immediate.

Let χ_0 and χ_1 denote the indicator functions of the intervals $[0, r_0^d]$ and $[0, r_1^d]$ respectively. Then

$$\exp(-2\langle\xi_n, \phi\rangle) \leq (\exp(2H\langle\xi_n, \chi_1\rangle))\mathbf{1}_{\{\langle\xi_n, \chi_0\rangle = 0\}}.$$

Let $A_{i,j}$ be the event that $n^2|X_i^n - X_j^n|^d \le r_1^d$. Let $C = (2r_1/r_0)^d$. If $\langle \xi_n, \chi_0 \rangle = 0$, then no ball of radius $n^{-2/d}r_1$ can contain more than C of the points $X_1^n, \ldots, X_n^n$, so that $\langle \xi_n, \chi_1 \rangle \le C\sum_{j=2}^n \mathbf{1}_{\{\cup_{i<j} A_{i,j}\}}$. Hence,

$$\exp(-2\langle \xi_n, \phi \rangle) \le \exp\left(2CH\sum_{j=2}^n \mathbf{1}_{\{\cup_{i<j} A_{i,j}\}}\right)$$

$$\le \prod_{j=2}^n (e^{2CH}\mathbf{1}_{\{\cup_{i<j} A_{i,j}\}} + 1).$$

There is a constant C' such that $\mu_n[A_{i,j}|X_1, \ldots, X_{j-1}] \le C'j/n^2$ for each j, and so by successive conditioning under μ_n,

$$\mathrm{E}\exp(-2\langle \xi_n, \phi \rangle) \le \prod_{j=2}^n \left(\frac{C'e^{2CH}}{n} + 1\right) \le \exp(C'e^{2CH}),$$

so that the random variables $\exp(-\langle \xi_n, \phi \rangle)$ are uniformly integrable. Then by (3.4),

$$\mathrm{E}(\exp -\langle \xi_n, \phi \rangle) \to \mathrm{E}(\exp -\langle \xi, \phi \rangle).$$

The same argument applies when ϕ is replaced by $\phi + g$, with $g \in C_K^+(\mathbb{R}_+)$. Thus the proof of Proposition 2.1 applies as in the proof of Theorem 3.1.

Remark. The practical significance of Theorem 3.1 is that pairwise interaction processes with $\phi \ge 0$ are a wide and useful class of models for spatial point processes with inhibition between points; for discussion see [2, 5, 15]. For these processes, Theorem 3.1 is useful whenever one needs to calculate something based only on the interpoint distances.

In the attractive case ($\phi \le 0$), matters are less clear. Empirical studies, for example by Geyer and Thompson [6], and Møller [9], have been made of the attractive Strauss model, in which $\phi \equiv \theta\mathbf{1}_{(0,\rho]}$ with parameters $\theta < 0$ and $\rho > 0$, and the underlying density f uniform on a given region such as the unit torus. These indicate that with n, ρ and λ fixed, as θ becomes more negative, there is a sudden transition from a Poisson-like distribution of points to one favouring highly clustered configurations. This limits the usefulness of the process as a model for moderately clustered configurations.

It can be shown that in the sparse limit for the attractive Strauss model with *any* negative θ, as n becomes large the distribution strongly favours highly clustered configurations. If we add a hard-core component to the potential, Theorem 3.2 shows that this theoretical difficulty is avoided; however, practitioners have found that some of the practical limitations alluded to in the previous paragraph remain. See the discussion in [9].

We conclude this section with a result on the *rate* of convergence in a special case of Theorem 3.1. For the rest of this section, take $\phi \equiv \theta\mathbf{1}_{(0,\rho]}$ with $\theta > 0$ and $\rho > 0$ (the inhibitive Strauss model). Instead of the entire point process ξ_n, consider the number of points of ξ_n in a fixed interval $(0, a]$, here denoted by $\xi_n(a)$. Theorem 3.1 shows that the distribution of $\xi_n(a)$ converges to the Poisson with mean $\lambda_0((0, a])$.

Write $\mathrm{Po}(\lambda)$ for a Poisson variable with mean λ. Given integer-valued random variables Y, Z, the total variation distance $D_{\mathrm{TV}}(Y, Z)$ between their distributions is $\sup |\mathrm{E}h(Y) - \mathrm{E}h(Z)|$ with the supremum taken over all test functions $h : \mathbf{Z} \to [0, 1]$. The rate of convergence result

is that under suitable smoothness conditions on f (see below), there is a constant c, depending on a, θ, ρ, and the density f, such that

$$D_{\mathrm{TV}}(\xi_n(a), \mathrm{Po}(\lambda_0(0, a])) \leq cn^{-\min(1, 2/d)}. \tag{3.5}$$

We sketch a proof of (3.5). Suppose $X_1, X_2, \ldots$ are independent with common density f. For $a > 0$, set $U_{n,a} = \sum_{i < j \leq n} \mathbf{1}_{\{n^2|X_i - X_j|^d \leq a\}}$. Then for any test function $h : \mathbf{Z} \to [0, 1]$,

$$\mathrm{E}[h(\xi_n(a))] = \frac{\mathrm{E}[h(U_{n,a}) \exp(-\theta U_{n,\rho})]}{\mathrm{E}[\exp(-\theta U_{n,\rho})]}. \tag{3.6}$$

If the restriction of f to its support $\mathrm{supp}(f)$ is sufficiently smooth, and also the boundary of $\mathrm{supp}(f)$ is well-behaved, then for any $a > 0$ we have

$$\left| \mathrm{E}U_{n,a} - \tfrac{1}{2} c_d a \int f^2 \right| = O(n^{-\min(1, 2/d)}). \tag{3.7}$$

The extra condition required on f is that (3.7) holds (as well as f being bounded). For example, if f is uniform on the unit cube then (3.7) holds.

Using Theorem 1 of Arratia *et al.* [1] (see also [3]), it can be shown that

$$D_{\mathrm{TV}}(U_{n,a}, \mathrm{Po}(\mathrm{E}U_{n,a})) = O(n^{-1}).$$

Moreover, for $a < b$, Theorem 2 of [1] can be used to show that the total variation distance, between the *joint* distribution of $U_{n,a}$ and $U_{n,b} - U_{n,a}$, and the joint distribution of independent Poissons with means $\mathrm{E}U_{n,a}$ and $\mathrm{E}[U_{n,b} - U_{n,a}]$, is also $O(n^{-1})$. Using these facts along with (3.6) and (3.7), one can establish (3.5); we omit further details.

The above suggests that $n^{-\min(1, 2/d)}$ may be the rate of convergence for the general case of Theorem 3.1, and for subsequent results in this paper, but we do not have proofs.

From a practical point of view an explicit value for c in (3.5) would be useful. With this in mind we give a more explicit result for the special case of the inhibitive Strauss model taking f to be the uniform distribution on the unit torus with $d = 2$. In this case, defining $u = \exp(\tfrac{1}{2}\pi\rho(e^{-\theta} - 1))$, we have

$$D_{\mathrm{TV}}(\xi_n(a), \mathrm{Po}(\lambda_0(0, a])) \leq \left(\frac{8}{u}(\pi \max(a, \rho))^2 + \frac{4}{u^2}(\pi\rho)^2 + \lambda_0((0, a]) \right) n^{-1} + O(n^{-2}).$$

4. Several types of particles

We now generalize the setting of the last section by allowing more than one type of particle. Strauss [16] also worked on the multitype setting, with random sample sizes, using different methods from those used here. We assume that there are α types of particle, and the null probability density function for particles of type a is f_a, assumed bounded for each a. In what follows we assume that $n_a = n_a(n)$ are numbers satisfying $\sum_{a=1}^{\alpha} n_a = n$ and $\lim_{n \to \infty}(n_a/n) = \pi_a \in (0, 1)$, so that necessarily $\sum_a \pi_a = 1$. The energy function $\phi_{a,b}$ is defined for each $(a, b) \in \Delta$ where we set $\Delta = \{(a, b) \in \mathbf{Z}^2 : 1 \leq a \leq b \leq \alpha\}$.

Our main result on multitype pairwise interaction distributions (Theorem 4.1) says that the rescaled interpoint distances between type a and b particles converge to Poisson processes with rates $\lambda_{a,b} \exp(-\phi_{a,b}(u)) \, du$, which are independent for different pairs (a, b).

First we obtain a Poisson limit in the case of no interactions ($\phi_{a,b} = 0$ for all a, b). This is a multitype generalization of a result of Silverman and Brown [14, Theorem C] on U-statistics. We wish to record small interpoint distances separately for each pair of types of particle. This gives us a point process in the union of Δ disjoint copies of $\mathbb{R}_+$, or more precisely, in the space $\mathbb{R}_+ \times \Delta$, metrized as a subset of $\mathbb{R}^3$. Let $\Omega_n = \prod_{a=1}^{\alpha}((\mathbb{R}^d)^{n_a})$. A typical element of Ω_n is a vector x of the form $x = \prod_{a \leq \alpha}(x_1^a, x_2^a, \ldots, x_{n_a}^a)$, with each $x_i^a \in \mathbb{R}^d$. An element x of Ω_n gives rise to a realization of a point measure $\xi_n(x)$ on $\mathbb{R}_+ \times \Delta$ by

$$\xi_n(x) = \sum{}'\delta((n^2|x_i^a - x_j^b|^d, (a, b))) \tag{4.1}$$

with $\sum'$ meaning a sum taken over all $(a, b) \in \Delta$, $i \leq n_a$, $j \leq n_b$ with $i < j$ if $a = b$.

Proposition 4.1. *Let μ_n be the measure on Ω_n corresponding to n_a independent particles of type a with common density f_a for each a;*

$$\mu_n(\mathrm{d}x) = \prod_{a=1}^{\alpha} \prod_{i=1}^{n_a} (f_a(x_i^a) \, \mathrm{d}x_i^a). \tag{4.2}$$

Let X_n be a random element of Ω_n with distribution μ_n. Then the induced point process $\xi_n(X_n)$ converges weakly to a Poisson process on $\mathbb{R}_+ \times \Delta$ with mean measure

$$\lambda(\mathrm{d}u \times \{(a, b)\}) = \lambda_{a,b} \, \mathrm{d}u, \qquad u \in \mathbb{R}_+, \quad (a, b) \in \Delta$$

where we define $\lambda_{a,b} = c_d \pi_a \pi_b \int f_a f_b$ if $a \neq b$ and $\lambda_{a,b} = (1/2)c_d \pi_a^2 \int f_a^2$ if $a = b$.

Proof. Let χ denote the indicator function of a set in $\mathbb{R}_+ \times \Delta$ of the form $\cup_{a<b} A_{a,b} \times \{(a, b)\}$, with each $A_{a,b}$ a finite union of bounded intervals. Let $\chi_{a,b}$ be the indicator function of $A_{a,b}$ for each $(a, b) \in \Delta$. By a result of Kallenberg ([8, Theorem 2.3] or [11, Proposition 3.22]), it suffices to prove that for any χ of this form the variables $\langle \xi_n, \chi \rangle$ are asymptotically Poisson with mean $\sum_{a \leq b} \int_{A_{a,b}} \lambda_{a,b}(\mathrm{d}x)$. In the case where all but one of the $A_{a,b}$ are empty, this follows from Corollary 2 of [10], and the proof in general is by a similar method using random sample sizes. Let $N_a = N_a(n)$ be independent Poisson variables with $EN_a = n_a$. Let ξ_n'' be the point process given by the right-hand side of (4.1) with $\sum'$ replaced by a summation (denoted $\sum''$) over all $(a, b) \in \Delta$, $i \leq N_a$, $j \leq N_b$, with $i < j$ if $a = b$. Then

$$\langle \xi_n'', \chi \rangle = \sum{}''\chi_{a,b}(n^2|X_i^a - X_j^b|^d).$$

The combined labelled sample $\{(x_i, a) : i \leq N_a, a \leq \alpha\}$ forms a Poisson process in the space $\Omega_0 := \mathbb{R}^d \times \{1, 2, \ldots, \alpha\}$ with mean measure ν_n given by $\nu_n(\mathrm{d}x \times \{a\}) = n_a f_a(x) \, \mathrm{d}x$, and $\langle \xi_n'', \chi \rangle$ is equal to the sum of the values of a symmetric zero-one function h_n on $\Omega_0 \times \Omega_0$, summed over all distinct pairs of these Poisson points in Ω_0. If U and V are independent random elements of Ω_0 with distribution $n^{-1}\nu_n$, it is straightforward to show that

$$(n^2/2)\mathrm{E}h_n(U, V) \to \sum_{a \leq b} \lambda_{a,b} \int_0^{\infty} \chi_{a,b}(u) \, \mathrm{d}u. \tag{4.3}$$

By Theorem 3.2 of [7], it follows that $\langle \xi_n'', \chi \rangle$ is asymptotically Poisson with mean given by the right hand side of (4.3). Finally, it can be shown that $E|\xi_n''(A) - \xi_n'(A)| \to 0$ by arguments from [7, p. 1353] and [10, p. 59].

Theorem 4.1. *Suppose for each* $(a, b) \in \Delta$ *the function* $\phi_{a,b} : \mathbb{R}_+ \to (-\infty, \infty]$ *is almost everywhere continuous and is bounded below and has compact support. Suppose that for each* (a, b), *either* $\phi_{a,b}$ *is non-negative or both* $\phi_{a,a}$ *and* $\phi_{b,b}$ *have hard-core components, in the terminology of Theorem 3.2. Suppose* μ'_n *is a probability measure on* $\Omega_n = \prod_{a=1}^{\alpha}((\mathbb{R}^d)^{n_a})$ *given by*

$$\mu'_n(\mathrm{d}x) = z_n^{-1} \exp\{-\sum{}' \phi_{a,b}(n^2|x_i^a - x_j^b|^d)\}\mu_n(\mathrm{d}x) \tag{4.4}$$

with μ_n *given by (4.2), and* z_n *a normalizing constant. Then if* X'_n *is a random element of* Ω_n *with distribution* μ'_n, *the sequence of point processes* $\xi_n(X'_n)$ *given by (4.1) converges weakly to a Poisson process on* $\mathbb{R}_+ \times \Delta$ *with mean measure* λ' *given by*

$$\lambda'(\mathrm{d}u \times \{(a, b)\}) = \lambda_{a,b} \exp(-\phi_{a,b}(u)) \, \mathrm{d}u$$

with $\lambda_{a,b}$ *as given in Proposition 4.1.*

Proof. If $\phi : (\mathbb{R}_+ \times \Delta) \to \mathbb{R}$ is given by $\phi((u, (a, b))) = \phi_{a,b}(u)$, we have

$$\mu'_n(\mathrm{d}x) = z_n^{-1} \exp(-\langle \xi_n(x), \phi \rangle)\mu_n(\mathrm{d}x).$$

When all the $\phi_{a,b}$ are non-negative the result is immediate from Propositions 2.1 and 4.1. In the case where $\phi_{a,b}$ takes negative values it is assumed that there exists $r_0 > 0$ such that $\phi_{a,a}(u)$ and $\phi_{b,b}(u)$ are both equal to $+\infty$ on $u \le r_0^d$. The convergence of z_n and of Laplace functionals can now be established along the lines of the proof of Theorem 3.2.

5. The area-interaction process

For the area-interaction point process, we restrict our attention to one type of particle. The form of density given in (1.2) can be re-stated as a joint distribution of n d-dimensional variables of the form

$$P[(X_1^n, X_2^n, \dots, X_n^n) \in \mathrm{d}x] = z_n^{-1} w_n(x) \prod_{i=1}^{n}(f(x_i) \, \mathrm{d}x_i), \tag{5.1}$$

with $x = (x_1, \dots, x_n)$, weight function w_n given by

$$w_n(x) = \exp\left\{\gamma_n\left(\left(\sum_{i=1}^{n} V_{r_n}(x_i)\right) - V_{r_n}(x_1, \dots, x_n)\right)\right\}, \tag{5.2}$$

and z_n a normalizing constant. This is equivalent to (1.2) because $\sum_{i=1}^{n} V_{r_n}(x_i)$ is a constant, given n. We take limits as $n \to \infty$ with parameters r_n, γ_n given by

$$n^2 r_n^d = \beta > 0 \tag{5.3}$$

and

$$r_n^d \gamma_n = \alpha < 0, \tag{5.4}$$

with β and α fixed constants. Conditions (5.3) and (5.4) are natural analogues to the sparse limit $\phi_n(x) = \phi(n^2|x|^d)$ considered earlier for the pairwise-interaction process. The condition $\alpha < 0$ says that the process is *inhibitive*, configurations without interaction being favoured.

We introduce further notation. Let $B_r(x)$ denote the ball of radius r centred at x (so $V_r(x)$ is its volume), and let the volume of the intersection of $B_1(0)$ and $B_1(x)$ be denoted $\psi(|x|)$. Finally define the function $\psi_\beta : \mathbb{R}_+ \to \mathbb{R}$ by

$$\psi_\beta(r) = \psi((r/\beta)^{1/d}). \tag{5.5}$$

Theorem 5.1. *Let f be a bounded probability density function on $\mathbb{R}^d$. Suppose that for each n, $X_1^n, \ldots, X_n^n$ are d-dimensional random variables with joint distribution given by (5.1)– (5.4). Then the sequence of one-dimensional point processes $\xi_n := \sum_{i<j\leq n} \delta(n^2|X_i^n - X_j^n|^d)$ converges weakly to a Poisson process on $\mathbb{R}_+$ with mean measure λ_1 given by*

$$\lambda_1(\mathrm{d}u) = \tfrac{1}{2}c_d\left(\int f^2\right)\mathrm{e}^{-\alpha\psi_\beta(u)}\,\mathrm{d}u. \tag{5.6}$$

Proof. We compare the area-interaction process with a pairwise-interaction process given by the joint distribution

$$\tilde{\mathrm{P}}[(X_1^n, X_2^n, \ldots, X_n^n) \in \mathrm{d}x] = \tilde{z}_n^{-1}\tilde{w}_n(x)\,\mathrm{d}x, \tag{5.7}$$

with

$$\tilde{w}_n(x_1, \ldots, x_n) = \exp\left(\alpha \sum_{i<j\leq n} \psi_\beta(n^2|x_i - x_j|^d)\right).$$

Let $Y_1, Y_2, Y_3, \ldots$ be independent d-valued variables with common density f. The normalizing constants z_n and $\tilde{z}_n$ are given by

$$z_n = \mathrm{E}w_n(Y_1, \ldots, Y_n); \qquad \tilde{z}_n = \mathrm{E}\tilde{w}_n(Y_1, \ldots, Y_n).$$

Let T_n be the number of 'interacting triples' (i, j, k) with $1 \leq i < j < k < n$ and $|Y_i - Y_j| < 2r_n$, $|Y_j - Y_k| < 2r_n$, and $|Y_i - Y_k| < 2r_n$. Then $\mathrm{E}[T_n] \to 0$ by (5.3), so

$$\lim_{n\to\infty} \mathrm{P}[T_n > 0] = 0. \tag{5.8}$$

By inclusion–exclusion,

$$\left(\sum_{i=1}^n V_{r_n}(Y_i)\right) - V_{r_n}(Y_1, \ldots, Y_n) = \sum_{i<j\leq n} \mathrm{Vol}(B_{r_n}(Y_i) \cap B_{r_n}(Y_j)) \quad \text{on } \{T_n = 0\}.$$

Since $\mathrm{Vol}(B_{r_n}(x) \cap B_{r_n}(y)) = r_n^d\psi(|x - y|/r_n) = r_n^d\psi_\beta(n^2|x - y|^d)$ for any x and y,

$$\left(\sum_{i=1}^n V_{r_n}(Y_i)\right) - V_{r_n}(Y_1, \ldots, Y_n) = r_n^d \sum_{i<j\leq n} \psi_\beta(n^2|Y_i - Y_j|^d) \quad \text{on } \{T_n = 0\}.$$

Hence, on $\{T_n = 0\}$ we have $w_n(Y_1, \ldots, Y_n) = \tilde{w}_n(Y_1, \ldots, Y_n)$, and so for any $g : \mathbb{R}_+ \to [0, \infty]$ of compact support, if we define the point process ζ_n by

$$\zeta_n := \sum_{i<j\leq n} \delta(n^2|Y_i - Y_j|^d),$$

and write Y_1^n for $(Y_1, \ldots, Y_n)$, we obtain

$$|\mathrm{E}e^{-\langle \zeta_n, g \rangle} \tilde{w}_n(Y_1^n) - \mathrm{E}e^{-\langle \zeta_n, g \rangle} w_n(Y_1^n)| \le \mathrm{E}[e^{-\langle \zeta_n, g \rangle} |\tilde{w}_n(Y_1^n) - w_n(Y_1^n)| \mathbf{1}_{\{T_n > 0\}}], \quad (5.9)$$

and this tends to zero by (5.8), because $\tilde{w}_n$ and w_n are uniformly bounded by 1. In particular, setting $g \equiv 0$, we have $\tilde{z}_n - z_n \to 0$.

Let $\mathrm{E}, \tilde{\mathrm{E}}$ denote expectation under distributions P and $\tilde{\mathrm{P}}$ defined at (5.1) and (5.7) respectively. Then for any $g : \mathbb{R}_+ \to [0, \infty]$ of compact support, the point process

$$\xi_n := \sum_{i < j \le n} \delta(n^2 |X_i^n - X_j^n|^d)$$

satisfies

$$\mathrm{E} \exp(-\langle \xi_n, g \rangle) = z_n^{-1} \mathrm{E}[e^{\langle -\zeta_n, g \rangle} w_n(Y_1^n)],$$

and

$$\tilde{\mathrm{E}} \exp(-\langle \xi_n, g \rangle) = \tilde{z}_n^{-1} \mathrm{E}[e^{-\langle \zeta_n, g \rangle} \tilde{w}_n(Y_1^n)].$$

By the proof of Theorem 3.1, both $\tilde{z}_n$ and $\mathrm{E}[e^{-\langle \zeta_n, g \rangle} \tilde{w}_n(Y_1^n)]$ tend to positive finite limits as $n \to \infty$. Therefore by (5.9), $\mathrm{E}e^{\langle -\xi_n, g \rangle}$ and $\tilde{\mathrm{E}}e^{\langle -\xi_n, g \rangle}$ converge to the same limit. By Lemma 2.2 the point process ξ_n obeys the same weak convergence under P as under $\tilde{\mathrm{P}}$, that is, to a Poisson process with mean measure λ_1, by Theorem 3.1.

Remark. Since the original purpose of the area-interaction process [2] was as a more satisfactory alternative to the pairwise interaction process in the attractive case, it would be of interest to have an analogous result to Theorem 5.1 for the attractive case $\alpha > 0$. However, the methods used here do not appear to work for this case, since (5.9) need not tend to zero. As in the pairwise interaction case, the weight of configurations with all particles close together grows in an uncontrolled way. Presumably, by adding a hard-core constraint to the attractive area-interaction process one could recover a Poisson limit theorem, as in the pairwise interaction case.

Acknowledgement

References

[1] ARRATIA, R., GOLDSTEIN, L. AND GORDON, L. (1989). Two moments suffice for Poisson approximations: the Chen–Stein method. *Ann. Prob.* **7**, 9–25.

[2] BADDELEY, A. J. AND VAN LIESHOUT, M. N. M. (1995). Area-interaction point processes. *Ann. Inst. Statist. Math.* **47**, 601–619.

[3] BARBOUR, A. D. AND EAGLESON, G. K. (1983). Poisson approximation for some statistics based on exchangeable trials. *Adv. Appl. Prob.* **15**, 585–600.

[4] DALEY, D. J. AND VERE-JONES, D. (1988). *An Introduction to the Theory of Point Processes.* Springer, New York.

[5] DIGGLE, P. J., FIKSEL, T., GRABARNIK, P., OGATA, Y., STOYAN, D. AND TANEMURA, M. (1994). On parameter estimation for pairwise interaction point processes. *Int. Statist. Rev.* **62**, 99–117.

[6] GEYER, C. J. AND THOMPSON, E. A. (1995). Annealing Markov chain Monte Carlo with applications to ancestral inference. *J. Amer. Statist. Assoc.* **90**, 909–920.

[7] JAMMALAMADAKA, S. R. AND JANSON, S. (1986). Limit theorems for a triangular scheme of U-statistics with applications to inter-point distances. *Ann. Prob.* **14**, 1347–1358.

[8] KALLENBERG, O. (1973). Characterization and convergence of random measures and point processes, *Z. Wahrscheinlichkeitsth.* **27**, 9–21.

[9] MØLLER, J. (1999). Markov chain Monte Carlo and spatial point processes. In *Proc. Seminaire Européen de Statistique Toulouse 1996 'Stochastic Geometry: Theory and Applications'*, ed. W. S. Kendall. CRC/Chapman and Hall, New York, pp. 141–172.

[10] PENROSE, M. D. (1995). Generalized two-sample *U*-statistics and a two-species reaction-diffusion model. *Stoch. Proc. Appl.* **55**, 57–64.

[11] RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.

[12] SAUNDERS, R. AND FUNK, G. M. (1977). Poisson limits for a clustering model of Strauss. *J. Appl. Prob.* **14**, 776–784.

[13] SAUNDERS, R., KRYSCIO, R. J. AND FUNK, G. M. (1982). Poisson limits for a hard-core clustering model. *Stoch. Proc. Appl.* **12**, 97–106.

[14] SILVERMAN, B. AND BROWN, T. (1978). Short distances, flat triangles and Poisson limits. *J. Appl. Prob.* **15**, 815–825.

[15] STOYAN, D., KENDALL, W. S. AND MECKE, J. (1995). *Stochastic Geometry and its Applications*, 2nd edn, John Wiley, Chichester, UK.

[16] STRAUSS, D. (1986). On a general class of models for interaction. *SIAM Rev.* **28**, 513–527.

[17] WIDOM, B. AND ROWLINSON, J. S. (1970). New model for the study of liquid–vapor phase transitions. *J. Chem. Phys.* **52**, 1670–1684.